Bergman kernels on generalized Hua domains*

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Abstract The Bergman kernel functions with explicit formulas of the generalized Hua domains are obtained. And the holomorphic automorphism group for each generalized Hua domain is also given.

Keywords: **generalized Hua domain, Bergman kernels, holomorphic automorphism group.**

Hua[1] obtained Bergman kernel functions with explicit formulas for four types of Cartan domains by using holomorphic transitive group (this method is called Hua method). For the two exceptional Cartan domains Yin[2] got their Bergman kernels in explicit forms. For bounded non-symmetric homogeneous domains, the explicit formulas of their Bergman kernel functions can also be obteined by Hua's method[3,4]. The explicit formulas of the Bergman kernel functions for "egg" domains can be obtained by summing an infinite series in some cases[5~9]. By now, one can compute the explicit formulas of Bergman kernel functions only for the above two types of domains. In general, it is difficult to get the domain whose Bergman kernel function can be gotten explicitly.

Since 1999, Yin has introduced four types of Hua domains. All of their Bergman kernel functions have been computed explicitly[10]. Recently Yin constructed the generalized Hua domains, which can be written as

$$GHE_{\mathrm{I}}(N_1,\cdots,N_r;m,n;p_1,\cdots,p_r;k)$$
$$=\{w_j\in C^{N_j},Z\in\Re_{\mathrm{I}}(m,n):\sum_{j=1}^{r}\|w_j\|^{2p_j}<\det(I-ZZ^{\mathrm{T}})^k,j=1,\cdots,r\},$$

$$GHE_{\mathrm{II}}(N_1,\cdots,N_r;p;p_1,\cdots,p_r;k)$$
$$=\{w_j\in C^{N_j},Z\in\Re_{\mathrm{II}}(p):\sum_{j=1}^{r}\|w_j\|^{2p_j}<\det(I-Z\bar{Z})^k,j=1,\cdots,r\},$$

$$GHE_{\mathrm{III}}(N_1,\cdots,N_r;q;p_1,\cdots,p_r;k)$$
$$=\{w_j\in C^{N_j},Z\in\Re_{\mathrm{III}}(q):\sum_{j=1}^{r}\|w_j\|^{2p_j}<\det(I+Z\bar{Z})^k,j=1,\cdots,r\},$$

$$GHE_{\mathrm{IV}}(N_1,\cdots,N_r;n;p_1,\cdots,p_r;k)$$
$$=\{w_j\in C^{N_j},Z\in\Re_{\mathrm{IV}}(n):\sum_{j=1}^{r}\|w_j\|^{2p_j}<(1+|ZZ'|^2-2ZZ^{\mathrm{T}})^k,j=1,\cdots,r\},$$

where $w_j=(w_{j1},\cdots,w_{jN_j})$, $j=1,\cdots,r$. $\Re_{\mathrm{I}}(m,n)$, $\Re_{\mathrm{II}}(p)$, $\Re_{\mathrm{III}}(q)$ and $\Re_{\mathrm{IV}}(n)$ denote the four types of Cartan domains in the sense of Hua[11] respectively. Z^{T} denotes the conjugate and transposed matrix of Z. $N_1,\cdots,N_r$ are positive integers, p_1, $p_2,\cdots,p_r,k$ are positive real numbers.

When $k=1$, the generalized Hua domains are the Hua domains.

In this paper, we only give the process of computing the Bergman kernel function for the first type of generalized Hua domain in the case that $p_1^{-1},\cdots,p_{r-1}^{-1}$ are positive integers and p_r, k are positive real numbers. For the other three types, because the methods used are the same as for the first type, we only give the final results here. The four types of the generalized Hua domains for $N_1=N_2=\cdots=N_r=1$ are denoted by GHE_{I}, GHE_{II}, GHE_{III} and GHE_{IV} respectively.

1 Preliminaries

Proposition 1[1]. (i) $Z\in\Re_{\mathrm{I}}(m,n)$, $\lambda>-1$, $J_{m,n}(\lambda)=\int_{\Re_{\mathrm{I}}(m,n)}\det(I-ZZ^{\mathrm{T}})^{\lambda}\mathrm{d}Z$, then

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$$J_{m,n}(\lambda) = \pi^{mn} \frac{\prod_{j=1}^{n}\Gamma(\lambda + j)\prod_{v=1}^{m}\Gamma(\lambda + v)}{\prod_{l=1}^{m+n}\Gamma(\lambda + l)}.$$

(ii) $Z \in \mathfrak{R}_{\text{II}}(p)$, $\lambda > -1$, $J_p(\lambda) = \int_{\mathfrak{R}_{\text{II}}(p)} \det(I - Z\bar{Z})^\lambda \mathrm{d}Z$, then

$$J_p(\lambda) = \pi^{\frac{p(p+1)}{2}} \frac{\prod_{j=2}^{p}\Gamma(2\lambda + 2j - 1)}{\prod_{v=1}^{p}(\lambda + v)\prod_{l=2}^{p}\Gamma(2\lambda + l + p)}.$$

(iii) $q \geqslant 2$, $Z \in \mathfrak{R}_{\text{III}}(q)$, $\lambda > -\frac{1}{2}$, $J_q(\lambda) = \int_{\mathfrak{R}_{\text{III}}(q)} \det(I + Z\bar{Z})^\lambda \mathrm{d}Z$, then

$$J_q(\lambda) = \pi^{\frac{q(q-1)}{2}} \frac{\prod_{j=2}^{q}\Gamma(2\lambda + 2j - 3)}{\prod_{l=2}^{q}\Gamma(2\lambda + l + q - 2)}.$$

(iv) $Z \in \mathfrak{R}_{\text{IV}}(n)$, $\beta(Z, Z) = 1 + |ZZ'|^2 - 2ZZ^{\text{T}}$, $\lambda > -1$, $J_n(\lambda) = \int_{\mathfrak{R}_{\text{IV}}(n)} [\beta(Z, Z)]^\lambda \mathrm{d}Z$, then

$$J_n(\lambda) = \pi^n \frac{\Gamma(\lambda + 1)}{2^{n-1}(2\lambda + n)\Gamma(\lambda + n)}.$$

Theorem 1. (i) The holomorphic automorphism group of GHE_{I} includes the following mappings (the set of such mappings is denoted by Aut(GHE_{I})):

$$\begin{cases} w_j^* = w_j \det(I - Z_0 Z_0^{\text{T}})^{\frac{k}{2p_j}} \det(I - ZZ_0^{\text{T}})^{-\frac{k}{p_j}}, \\ \qquad 1 \leqslant j \leqslant r; \\ Z^* = A(Z - Z_0)(I - Z_0^{\text{T}}Z)^{-1}D^{-1}, \end{cases} \tag{1}$$

where $A^{\text{T}}A = (I - Z_0Z_0^{\text{T}})^{-1}$, $D^{\text{T}}D = (I - Z_0^{\text{T}}Z_0)^{-1}$, $Z_0 \in \mathfrak{R}_{\text{I}}(m, n)$, $Z \in \mathfrak{R}_{\text{I}}(m, n)$. This mapping maps $(w, Z_0) \in GHE_{\text{I}}$ onto $(w^*, 0)$.

(ii) The holomorphic automorphism group of GHE_{II} includes the following mappings (the set of such mappings is denoted by Aut(GHE_{II})):

$$\begin{cases} w_j^* = w_j \det(I - Z_0 \overline{Z_0})^{\frac{k}{2p_j}} \det(I - Z\overline{Z_0})^{-\frac{k}{p_j}}, \\ \qquad 1 \leqslant j \leqslant r; \\ Z^* = A(Z - Z_0)(I - \overline{Z_0}Z)^{-1}\bar{A}^{-1}, \end{cases}$$

where $A^{\text{T}}A = (I - Z_0\overline{Z_0})^{-1}$, $Z_0 \in \mathfrak{R}_{\text{II}}(p)$, $Z \in \mathfrak{R}_{\text{II}}(p)$. This mapping maps $(w, Z_0) \in GHE_{\text{II}}$ onto $(w^*, 0)$.

(iii) The holomorphic automorphism group of GHE_{III} includes the following mappings (the set of such mappings is denoted by Aut(GHE_{III})):

$$\begin{cases} w_j^* = w_j \det(I + Z_0 \overline{Z_0})^{\frac{k}{2p_j}} \det(I + Z\overline{Z_0})^{-\frac{k}{p_j}}, \\ \qquad 1 \leqslant j \leqslant r; \\ Z^* = A(Z - Z_0)(I + \overline{Z_0}Z)^{-1}\bar{A}^{-1}, \end{cases}$$

where $A^{\text{T}}A = (I + Z_0\overline{Z_0})^{-1}$, $Z_0 \in \mathfrak{R}_{\text{III}}(q)$, $Z \in \mathfrak{R}_{\text{III}}(q)$. This mapping maps $(w, Z_0) \in GHE_{\text{III}}$ onto $(w^*, 0)$.

(iv) The holomorphic automorphism group of GHE_{IV} includes the following mappings (the set of such mappings is denoted by Aut(GHE_{IV})):

$$\begin{cases} w_j^* = w_j e^{\sqrt{-1}\theta_j} B^{-\frac{k}{p_j}}, \\ \qquad 1 \leqslant j \leqslant r; \\ Z^* = B^{-1}\left[Z - \left(\frac{1 + ZZ'}{2}, \frac{1 - ZZ'}{2\sqrt{-1}}\right) X_0\right]D, \end{cases}$$

where

$$AA' = (I - X_0X_0')^{-1}, \quad DD' = (I - X_0'X_0)^{-1},$$
$$Z_0 \in \mathfrak{R}_{\text{IV}}(n), \quad Z \in \mathfrak{R}_{\text{IV}}(n),$$

$$X_0 = \frac{-1}{1 - |Z_0Z_0'|^2} \cdot \begin{pmatrix} (\overline{Z_0Z_0'} - 1)Z_0 + (Z_0Z_0' - 1)\overline{Z_0} \\ \sqrt{-1}(Z_0Z_0' + 1)\overline{Z_0} - \sqrt{-1}(\overline{Z_0Z_0'} + 1)Z_0 \end{pmatrix},$$

$$B = \left[\left(\frac{1 + ZZ'}{2}, \frac{1 - ZZ'}{2\sqrt{-1}}\right) - ZX_0'\right]A\begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}.$$

This mapping maps $(w, Z_0) \in GHE_{\text{IV}}$ onto $(w^*, 0)$.

Proof. It is well known that $Z^* = A(Z - Z_0) \cdot (I - Z_0^{\text{T}}Z)^{-1}D^{-1}$ is the holomorphic automorphism of $\mathfrak{R}_{\text{I}}(m, n)$[11]. Hence, we can check that

$$\begin{aligned} &(\det(I - Z^*Z^{*\text{T}}))^k - \sum_{j=1}^{r} |w_j^*|^{2p_j} \\ &= (\det(I - Z_0Z_0^{\text{T}}))^k |\det(I - ZZ_0^{\text{T}})|^{-2k} \\ &\quad \cdot \left((\det(I - ZZ^{\text{T}}))^k - \sum_{j=1}^{r} |w_j|^{2p_j}\right). \end{aligned}$$

Therefore, the mappings given by (1) are the holomorphic automorphisms of GHE_{I}.

Theorem 2. (i) Let $x_j = x_j(w, Z) = |w_j|^2 [\det(I - ZZ^{\text{T}})]^{-\frac{k}{p_j}}$, $j = 1, 2, \cdots, r$. Then $x_1, \cdots, x_r$ are Aut (GHE_{I})-invariants. i. e $x_j(w^*, Z^*) = x_j(w, Z)$, $j = 1, 2, \cdots, r$.

(ii) Let $x_j = x_j(w, Z) = |w_j|^2 [\det(I - Z$

$\overline{Z})]^{-\frac{k}{p_j}}$, $j = 1, 2, \cdots, r$. Then $x_1, \cdots, x_r$ are $\mathrm{Aut}(GHE_{\mathrm{II}})$-invariants.

(iii) Let $x_j = x_j(w, Z) = |w_j|^2[\det(I + Z\overline{Z})]^{-\frac{k}{p_j}}$, $j = 1, 2, \cdots, r$. Then $x_1, \cdots, x_r$ are $\mathrm{Aut}(GHE_{\mathrm{III}})$-invariants.

(iv) Let $x_j = x_j(w, Z) = |w_j|^2[\beta(Z, Z)]^{-\frac{k}{p_j}}$, $j = 1, 2, \cdots, r$. Then $x_1, \cdots, x_r$ are $\mathrm{Aut}(GHE_{\mathrm{IV}})$-invariants.

Proof. We know that

$$\begin{aligned} x_j(w^*, Z^*) &= |w_j^*|^2[\det(I - Z^*Z^{*\mathrm{T}})]^{-\frac{k}{p_j}} \\ &= |w_j|^2[\det(I - Z_0Z_0^{\mathrm{T}})]^{\frac{k}{p_j}} \\ &\quad \cdot |\det(I - ZZ_0^{\mathrm{T}})|^{-\frac{2k}{p_j}} \\ &\quad \cdot [\det(I - Z_0Z_0^{\mathrm{T}}) \\ &\quad \cdot |\det(I - ZZ_0^{\mathrm{T}})|^{-2} \\ &\quad \cdot \det(I - ZZ^{\mathrm{T}})]^{-\frac{k}{p_j}} \\ &= |w_j|^2[\det(I - ZZ^{\mathrm{T}})]^{-\frac{k}{p_j}} \\ &= x_j(w, Z). \text{ Q.E.D.} \end{aligned}$$

Proposition 2[10]. Let $P(x)$ be a polynomial of degree n in x: $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, then $P(x)$ can be rewritten as the following form:

$$\begin{aligned} P(x) &= b_n(x + n)(x + n - 1)\cdots(x + 1) \\ &\quad + b_{n-1}(x + n - 1)\cdots(x + 1) + \cdots \\ &\quad + b_1(x + 1) + b_0 \\ &= \sum_{j=0}^{n} b_j\Gamma(x + j + 1)/\Gamma(x + 1), \end{aligned}$$

where $b_n = a_n$, $b_0 = P(-1)$ and $P(-1) = b_0$, $P(-2) = -b_1 + b_0$,

$$\begin{aligned} P(-j-1) &= (-1)^j\frac{\Gamma(j+1)}{\Gamma(1)}b_j \\ &\quad + (-1)^{j-1}\frac{\Gamma(j)}{\Gamma(2)}b_{j-1} + \cdots + b_0, \end{aligned}$$

then we have

$$\begin{aligned} b_j &= \left[P(-j-1) - \sum_{k=0}^{j-1}\frac{(-1)^k\Gamma(j+1)b_k}{\Gamma(j-k+1)}\right] \Big/ \\ &\quad \cdot [(-1)^j\Gamma(j+1)]. \end{aligned}$$

This is the recurrence formula, from $b_0, b_1, \cdots, b_{j-1}$, we can get b_j.

Proposition 3[9]. Let

$$\begin{aligned} &h_s(x_1, \cdots, x_{N-1}) \\ &\quad = \sum_{(j_1, \cdots, j_{N-1})}\frac{\Gamma\left(s + \sum_{l=1}^{N-1}\frac{j_l+1}{p_l}\right)}{\prod_{l=1}^{N-1}\Gamma\left(\frac{j_l+1}{p_l}\right)}x_1^{j_1}x_2^{j_2}\cdots x_{N-1}^{j_{N-1}}, \end{aligned}$$

where $s > 0$, $\frac{1}{p_l}(1 \leqslant l \leqslant N-1)$ are positive integers, $\sum_{l=1}^{N-1}|x_l|^{p_l} < 1$. Then

$$\begin{aligned} h_s(x_1, \cdots, x_{N-1}) &= \frac{\partial^{N-1}}{\partial x_1\cdots\partial x_{N-1}} \\ &\quad \cdot \left[\sum_{v_1=0}^{q_1-1}\cdots\sum_{v_{N-1}=0}^{q_{N-1}-1}\frac{\Gamma(s)}{\left(1 - \sum_{l=1}^{N-1}\omega_l^{v_l}x_l^{\frac{1}{q_l}}\right)^s}\right], \end{aligned}$$

where $q_l = \frac{1}{p_l}$, $\omega_l = \mathrm{e}^{\frac{2\pi}{q_l}\sqrt{-1}}$, $x_l^{\frac{1}{q_l}} = |x_l|^{\frac{1}{q_l}}\mathrm{e}^{\frac{\phi_l}{q_l}\sqrt{-1}}$, $-\pi < \phi_l = \arg x_l < \pi$, $1 \leqslant l \leqslant N-1$.

2 Bergman kernel function

Let $(w^*, Z^*) = f(w, Z) \in \mathrm{Aut}(GHE_{\mathrm{I}})$ which maps (w, Z_0) onto $(w^*, 0)$. Let $K_{\mathrm{I}}(w, Z; \overline{w}, \overline{Z})$ be the Bergman kernel function for GHE_{I}. Due to the well known transformation formula on Bergman kernel[10], we have $K_{\mathrm{I}}(w, Z; \overline{w}, \overline{Z}) = K_{\mathrm{I}}(w^*, 0; \overline{w^*}, 0)\left|\det\left(\frac{\partial(w^*, 0)}{\partial(w, Z)}\right)\right|^2_{Z_0 = Z}$. But

$$\left(\frac{\partial(w^*, Z^*)}{\partial(w, Z)}\right) = \begin{pmatrix} \frac{\partial w^*}{\partial w} & 0 \\ * & \frac{\partial Z^*}{\partial Z} \end{pmatrix}_{Z_0 = Z}.$$

Therefore, we have

$$\begin{aligned} &\left|\det\left(\frac{\partial(w^*, Z^*)}{\partial(w, Z)}\right)\right|^2_{Z_0 = Z} \\ &\quad = \left|\det\left(\frac{\partial w^*}{\partial w}\right)\det\left(\frac{\partial Z^*}{\partial Z}\right)\right|^2_{Z_0 = Z} \\ &\quad = \prod_{j=1}^{r}\left|\det(I - Z_0Z_0^{\mathrm{T}})^{\frac{k}{2p_j}}\det(I - ZZ_0^{\mathrm{T}})^{-\frac{k}{p_j}}\right|^2 \\ &\qquad \cdot \det(I - ZZ^{\mathrm{T}})^{-(m+n)}|_{Z_0 = Z} \\ &\quad = \det(I - ZZ^{\mathrm{T}})^{-\left(m + n + \frac{k}{p_1} + \cdots + \frac{k}{p_r}\right)}. \end{aligned}$$

We will compute the $K_{\mathrm{I}}(w^*, 0; \overline{w}^*, 0)$ in the following. For simplicity, let w denote w^*.

Because GHE_{I} is a semi-Reinhardt domain, the complete orthonormal system of GHE_{I} consists of $\{w^jP_{vl}^{(j)}(Z)\}$[10], where $j = (j_1, \cdots, j_r)$, $j_1, j_2, \cdots, j_r = 0, 1, \cdots$; $v = 0, 1, \cdots$; $l = 1, 2, \cdots, m_v$; $m_v =$

$(mn+v-1)!\ [v!\ (mn-1)!]^{-1}$.

By the definition of the Bergman kernel function, we have

$$K_{\mathrm{I}}(w,0;\bar{w},0)=\sum_{|j|\geqslant 0,\, v\geqslant 0,\, l=1,2,\cdots,m_v}|w^jP_{vl}^{(j)}(0)|^2.$$

Because $P_{vl}^{(j)}(0)=0$ when $v\geqslant 1$ and $m_v=1$ when $v=0$, we have

$$K_{\mathrm{I}}(w,0;\bar{w},0)=\sum_{|j|\geqslant 0}|w^jP_{01}^{(j)}(0)|^2.$$

Because $P_{01}^{(j)}(Z)$ is a polynomial of degree 0, $P_{01}^{(j)}(Z)=P_{01}^{(j)}(0)$. $w^jP_{01}^{(j)}(0)$ belongs to the complete orthonormal system for GHE_{I}. Then

$$\int_{GHE_{\mathrm{I}}}|w|^{2j}|P_{01}^{(j)}(0)|^2\mathrm{d}w\mathrm{d}Z=1.$$

Hence $|P_{01}^{(j)}(0)|^{-2}=\int_{GHE_{\mathrm{I}}}|w|^{2j}\mathrm{d}w\mathrm{d}Z$.

By Proposition 1, we have

$$|P_{01}^{(j)}(0)|^{-2}=\frac{\pi^r}{\prod_{l=1}^r p_l}\cdot\frac{\prod_{l=1}^r\Gamma\left(\frac{j_l+1}{p_l}\right)}{\Gamma\left(1+\sum_{l=1}^r\frac{j_l+1}{p_l}\right)}\int_{\mathfrak{R}_{\mathrm{I}}(m,n)}\det(I-ZZ^{\mathrm{T}})^{\left(k\sum_{l=1}^r\frac{j_l+1}{p_l}\right)}\mathrm{d}Z$$

$$=\frac{\pi^{mn+r}}{\prod_{l=1}^r p_l}\cdot\frac{\prod_{l=1}^r\Gamma\left(\frac{j_l+1}{p_l}\right)}{\Gamma\left(1+\sum_{l=1}^r\frac{j_l+1}{p_l}\right)}\cdot\frac{\prod_{v=1}^m\Gamma\left(k\sum_{l=1}^r\frac{j_l+1}{p_l}+v\right)\prod_{v=1}^n\Gamma\left(k\sum_{l=1}^r\frac{j_l+1}{p_l}+v\right)}{\prod_{v=1}^{m+n}\Gamma\left(k\sum_{l=1}^r\frac{j_l+1}{p_l}+v\right)}.$$

Therefore

$$|P_{01}^{(j)}(0)|^2=C\frac{\Gamma(1+\lambda)}{\prod_{l=1}^r\Gamma\left(\frac{j_l+1}{p_l}\right)}\cdot\frac{\prod_{v=1}^{m+n}\Gamma(k\lambda+v)}{\prod_{v=1}^n\Gamma(k\lambda+v)\prod_{v=1}^m\Gamma(k\lambda+v)},$$

where

$$C=\frac{\prod_{l=1}^r p_l}{\pi^{mn+r}};\quad \lambda=\sum_{l=1}^r\frac{j_l+1}{p_l}.$$

But

$$\frac{\prod_{v=1}^{m+n}\Gamma(k\lambda+v)}{\prod_{v=1}^n\Gamma(\lambda+v)\prod_{v=1}^m\Gamma(k\lambda+v)}$$
$$=[(k\lambda+n)(k\lambda+n-1)\cdots(k\lambda+1)]$$
$$\cdot[(k\lambda+n+1)(k\lambda+n)\cdots(k\lambda+2)]\cdots$$
$$\cdot[(k\lambda+n+m-1)(k\lambda+n+m-2)\cdots$$
$$\cdot(k\lambda+m)]$$

is a polynomial of degree mn in λ and denoted by $f_{1k}(\lambda)$. According to Proposition 2, there exist $mn+1$ constants $b_j(k), j=0,1,\cdots,mn$, such that

$$\frac{\prod_{v=1}^{m+n}\Gamma(k\lambda+v)}{\prod_{v=1}^n\Gamma(k\lambda+v)\prod_{v=1}^m\Gamma(k\lambda+v)}$$

$$=\sum_{v=0}^{mn}b_v(k)\frac{\Gamma(\lambda+v+1)}{\Gamma(\lambda+1)},$$

where $b_0(k)=f_{1k}(-1)$, the other $b_j(k)$ are determined by the following recurrence formula:

$$b_j(k)=\frac{f_{1k}(-j-1)-\sum_{v=0}^{j-1}\frac{b_v(k)(-1)^v\Gamma(j+1)}{\Gamma(j-v+1)}}{(-1)^j\Gamma(j+1)},$$
$$j=1,2,\cdots,mn.$$

So

$$|P_{01}^{(j)}(0)|^2=C\cdot\frac{\sum_{v=0}^{mn}b_v(k)\Gamma\left(\sum_{l=1}^r\frac{j_l+1}{p_l}+v+1\right)}{\prod_{l=1}^r\Gamma\left(\frac{j_l+1}{p_l}\right)}.$$

Therefore, we have

$$K_{\mathrm{I}}(w,0;\bar{w},0)$$
$$=\sum_{|j|\geqslant 0}\sum_{v=0}^{mn}C\frac{b_v(k)\Gamma\left(\sum_{l=1}^r\frac{j_l+1}{p_l}+v+1\right)}{\prod_{l=1}^r\Gamma\left(\frac{j_l+1}{p_l}\right)}\cdot|w_1|^{2j_1}\cdots|w_r|^{2j_r}$$
$$=C\sum_{v=0}^{mn}b_v(k)\sum_{j_r=0}^{\infty}\frac{|w_r|^{2j_r}}{\Gamma\left(\frac{j_r+1}{p_r}\right)}\cdot\sum_{(j_1\cdots j_{r-1})}\frac{\Gamma\left(\sum_{l=1}^{r-1}\frac{j_l+1}{p_l}+\frac{j_r+1}{p_r}+v+1\right)}{\prod_{l=1}^{r-1}\Gamma\left(\frac{j_l+1}{p_l}\right)}\cdot|w_1|^{2j_1}\cdots|w_{r-1}|^{2j_{r-1}}.$$

Let $s=v+1+\frac{j_r+1}{p_r}$, $x_l=|w_l|^2$, $1\leqslant l\leqslant r$, it is obvious that $\sum_{l=1}^{r-1} |x_l|^{p_l} < 1$ when $Z=0$. It follows from Proposition 3 that

$$K_{\mathrm{I}}(w,0;\bar{w},0) = C\sum_{v=0}^{mn} b_v(k) \sum_{j_r=0}^{\infty} h_s(x_1,x_2,\cdots,x_{r-1}) \frac{x_r^{j_r}}{\Gamma\left(\frac{j_r+1}{p_r}\right)}$$

$$= C\sum_{v=0}^{mn} b_v(k) \frac{\partial^{r-1}}{\partial x_1 \partial x_2 \cdots \partial x_{r-1}} \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_{r-1}=0}^{q_{r-1}-1} \sum_{j_r=0}^{\infty} \frac{\Gamma\left(1+v+\frac{j_r+1}{p_r}\right) \cdot x_r^{j_r}}{\Gamma\left(\frac{j_r+1}{p_r}\right) \cdot \left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{1+v+\frac{j_r+1}{p_r}}},$$

where $q_l = \frac{1}{p_l}$, $\omega_l = e^{\frac{2\pi}{q_l}\sqrt{-1}}$, $1 \leqslant l \leqslant r-1$. Because

$$\frac{\Gamma\left(1+v+\frac{j_r+1}{p_r}\right)}{\Gamma\left(\frac{j_r+1}{p_r}\right)} = \left(\frac{j_r+1}{p_r}+v\right)\left(\frac{j_r+1}{p_r}+v-1\right)\cdots \cdot \left(\frac{j_r+1}{p_r}+1\right)\left(\frac{j_r+1}{p_r}\right)$$

$$= \frac{1}{p_r^{v+1}}(j_r+1+p_r v) \cdot (j_r+1+p_r(v-1)) \cdots \cdot (j_r+1+p_r)(j_r+1)$$

is a polynomial of degree $v+1$ in j_r. According to Proposition 2, there exist $v+2$ constants c_{vj}, $j=0,1,\cdots,v+1$ such that $g_{1v}(j_r) := \frac{\Gamma\left(1+v+\frac{j_r+1}{p_r}\right)}{\Gamma\left(\frac{j_r+1}{p_r}\right)} = \sum_{j=0}^{v+1} c_{vj} \frac{\Gamma(j_r+j+1)}{\Gamma(j_r+1)}$ and $c_{v0}=0$, the other c_{vj} are determined by the following recurrence formula:

$$c_{vj} = \frac{g_{1v}(-j-1) - \sum_{l=0}^{j-1} \frac{c_{vl}(-1)^l \Gamma(j+1)}{\Gamma(j-l+1)}}{(-1)^j \Gamma(j+1)}, \quad j = 1,2,\cdots,v+1. \tag{2}$$

Therefore, let

$$t = \frac{x_r}{\left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{\frac{1}{p_r}}}. \tag{3}$$

Then, we have (it is similar to that described in Ref. [10])

$$\sum_{j_r=0}^{\infty} \frac{\Gamma\left(1+v+\frac{j_r+1}{p_r}\right)}{\Gamma\left(\frac{j_r+1}{p_r}\right)} \cdot \frac{x_r^{j_r}}{\left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{1+v+\frac{j_r+1}{p_r}}}$$

$$= \left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{-\left(v+1+\frac{1}{p_r}\right)} \sum_{j=1}^{v+1} c_{vj} \frac{\Gamma(j+1)}{(1-t)^{j+1}}.$$

Let w become w^*, we have

$$K_{\mathrm{I}}(w^*,0;\bar{w}^*,0) = C\sum_{v=0}^{mn} \frac{b_v(k)\partial^{r-1}}{\partial x_1 \partial x_2 \cdots \partial x_{r-1}} \cdot \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_{r-1}=0}^{q_{r-1}-1} \sum_{j=1}^{v+1} \frac{c_{vj}\Gamma(j+1)(1-t)^{-(j+1)}}{\left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{1+v+\frac{1}{p_r}}}$$

$$= C \frac{\partial^{r-1}}{\partial x_1 \partial x_2 \cdots \partial x_{r-1}} \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_{r-1}=0}^{q_{r-1}-1} \sum_{v=0}^{mn} \sum_{j=1}^{v+1} b_v(k) c_{vj} \cdot \frac{\Gamma(j+1)(1-t)^{-(j+1)}}{\left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{1+v+\frac{1}{p_r}}},$$

where $x_l = |w_l^*|^2 = |w_l|^2 \det(I - ZZ^{\mathrm{T}})^{-\frac{k}{p_l}}$, $1 \leqslant l \leqslant r$, $q_l = \frac{1}{p_l}$, $\omega_l = e^{\frac{2\pi}{q_l}\sqrt{-1}}$, $1 \leqslant l \leqslant r-1$, t is determined by (3).

So

$$K_{\mathrm{I}}(w,Z;\bar{w},\bar{Z}) = K_{\mathrm{I}}(w^*,0;\bar{w}^*,0)\det(I - ZZ^{\mathrm{T}})^{-\left(m+n+\sum_{l=1}^{r}\frac{k}{p_l}\right)}$$

$$= C \frac{\partial^{r-1}}{\partial x_1 \partial x_2 \cdots \partial x_{r-1}} \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_{r-1}=0}^{q_{r-1}-1} \sum_{v=0}^{mn} \sum_{j=1}^{v+1} \cdot \frac{\Gamma(j+1) b_v(k) c_{vj} \det(I - ZZ^{\mathrm{T}})^{-\left(m+n+\sum_{l=1}^{r}\frac{k}{p_l}\right)}}{(1-t)^{(j+1)}\left(1 - \sum_{l=1}^{r-1} \omega_l^{v_l} x_l^{\frac{1}{q_l}}\right)^{1+v+\frac{1}{p_r}}}.$$

If we consider w_j of GHE_{I} as $w_j = (w_{j1}, \cdots, w_{jN_j})$, $j=1,2,\cdots,r$, by using the principle of inflation[10], the Bergman kernel function on $GHE_{\mathrm{I}}(N_1, N_2, \cdots, N_r; m, n; p_1, p_2, \cdots, p_r; k)$ is:

$$K_{G\mathrm{I}}(w,Z;\bar{w},\bar{Z}) = \frac{\prod_{l=1}^{r} p_l}{\pi^{\left(mn+\sum_{l=1}^{r} N_l\right)}} \frac{\partial^{N_1+N_2+\cdots+N_{r-1}+N_{r-1}}}{\partial x_1^{N_1} \partial x_2^{N_2} \cdots \partial x_{r-1}^{N_{r-1}} \partial x_r^{N_r-1}} \cdot \sum_{v_1=0}^{q_1-1} \cdots \sum_{v_{v-1}=0}^{q_{r-1}-1} \sum_{v=0}^{mn} \sum_{j=1}^{v+1} \frac{\Gamma(j+1)}{(1-t)^{j+1}}$$

$$\cdot \frac{b_v(k)c_{vj}\det(I-ZZ^{\mathrm{T}})^{-\left(m+n+\sum_{l=1}^{r}\frac{kN_l}{p_l}\right)}}{\left(1-\sum_{l=1}^{r-1}\omega_l^{v_l}x_l^{\frac{1}{q_l}}\right)^{v+1+\frac{1}{p_r}}},$$

where $|w_l|^2=|w_{l1}|^2+\cdots+|w_{lN_l}|^2$, $l=1,\cdots,r$; x_l, t etc. as before.

Similarly, we can get the Bergman kernel functions with explicit formulas for the other three types of the generalized Hua domains in the following.

The Bergman kernel function of the generalized Hua domain of the second type $GHE_{\mathrm{II}}(N_1,N_2,\cdots,N_r;p;p_1,p_2,\cdots,p_r;k)$ is

$$K_{G\mathrm{II}}(w,Z;\overline{w},\overline{Z})$$
$$=\frac{\prod_{l=1}^{r}p_l}{\pi^{\left(\frac{p(p+1)}{2}+\sum_{l=1}^{r}N_l\right)}}\frac{\partial^{N_1+N_2+\cdots+N_{r-1}+N_r-1}}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_{r-1}^{N_{r-1}}\partial x_r^{N_r-1}}$$
$$\cdot\sum_{v_1=0}^{q_1-1}\cdots\sum_{v_{r-1}=0}^{q_{r-1}-1}\sum_{v=0}^{\frac{p(p+1)}{2}}\sum_{j=1}^{v+1}\frac{\Gamma(j+1)}{(1-t)^{j+1}}$$
$$\cdot\frac{b_v(k)c_{vj}\det(I-Z\overline{Z})^{-\left(p+1+\sum_{l=1}^{r}\frac{kN_l}{p_l}\right)}}{\left(1-\sum_{l=1}^{r-1}\omega_l^{v_l}x_l^{\frac{1}{q_l}}\right)^{v+1+\frac{1}{p_r}}},$$

where $|w_l|^2=|w_{l1}|^2+\cdots+|w_{lN_l}|^2$; $x_l=|w_l|^2\det(I-Z\overline{Z})^{-\frac{k}{p_l}}$, $1\leqslant l\leqslant r$; $q_l=\frac{1}{p_l}$, $\omega_l=e^{\frac{2\pi}{q_l}\sqrt{-1}}$, $1\leqslant l\leqslant r-1$; t is determined by (3). c_{vj}, $0\leqslant v\leqslant\frac{p(p+1)}{2}$, $1\leqslant j\leqslant v+1$ are determined by (2). Let

$$f_{2k}(\lambda):=\frac{\prod_{v=1}^{p}(k\lambda+v)\prod_{l=2}^{p}\Gamma(2k\lambda+l+p)}{\prod_{j=2}^{p}\Gamma(2k\lambda+2j-1)},$$

then $b_0(k)=f_{2k}(-1)$,

$$b_j(k)=\frac{f_{2k}(-j-1)-\sum_{v=0}^{j-1}\frac{b_v(k)(-1)^v\Gamma(j+1)}{\Gamma(j-v+1)}}{(-1)^j\Gamma(j+1)},$$
$$j=1,2,\cdots,\frac{p(p+1)}{2}.$$

The Bergman kernel function of the generalized Hua domain of the third type $GHE_{\mathrm{III}}(N_1,N_2,\cdots,N_r;q;p_1,p_2,\cdots,p_r;k)$ is

$$K_{G\mathrm{III}}(w,Z;\overline{w},\overline{Z})$$
$$=\frac{\prod_{l=1}^{r}p_l}{\pi^{\left(\frac{q(q-1)}{2}+\sum_{l=1}^{r}N_l\right)}}\frac{\partial^{N_1+N_2+\cdots+N_{r-1}+N_{r-1}}}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_{r-1}^{N_{r-1}}\partial x_r^{N_{r-1}}}$$
$$\cdot\sum_{v_1=0}^{q_1-1}\cdots\sum_{v_{r-1}=0}^{q_{r-1}-1}\sum_{v=0}^{\frac{q(q-1)}{2}}\sum_{j=1}^{v+1}\frac{\Gamma(j+1)}{(1-t)^{j+1}}$$
$$\cdot\frac{b_j(k)c_{vj}\det(I+Z\overline{Z})^{-\left(q-1+\sum_{l=1}^{r}\frac{kN_l}{p_l}\right)}}{\left(1-\sum_{l=1}^{r-1}\omega_l^{v_l}x_l^{\frac{1}{q_l}}\right)^{v+1+\frac{1}{p_r}}},$$

where $|w_l|^2=|w_{l1}|^2+\cdots+|w_{lN_l}|^2$; $x_l=|w_l|^2\det(I+Z\overline{Z})^{-\frac{k}{p_l}}$, $1\leqslant l\leqslant r$; $q_l=\frac{1}{p_l}$, $\omega_l=e^{\frac{2\pi}{q_l}\sqrt{-1}}$, $1\leqslant l\leqslant r-1$; t is determined by (3). c_{vj}, $0\leqslant v\leqslant\frac{q(q-1)}{2}$, $1\leqslant j\leqslant v+1$ are determined by (2). Let

$$f_{3k}(\lambda):=\frac{\prod_{l=2}^{q}\Gamma(2k\lambda+l+q-2)}{\prod_{j=2}^{q}\Gamma(2k\lambda+2j-3)},$$

then $b_0(k)=f_{3k}(-1)$,

$$b_j(k)=\frac{f_{3k}(-j-1)-\sum_{v=0}^{j-1}\frac{b_v(k)(-1)^v\Gamma(j+1)}{\Gamma(j-v+1)}}{(-1)^j\Gamma(j+1)},$$
$$j=1,2,\cdots,\frac{q(q-1)}{2}.$$

The Bergman kernel function of the generalized Hua domain of the fourth type $GHE_{\mathrm{IV}}(N_1,N_2,\cdots,N_r;n;p_1,p_2,\cdots,p_r;k)$ is

$$K_{G\mathrm{IV}}(w,Z;\overline{w},\overline{Z})$$
$$=\frac{\prod_{l=1}^{r}p_l}{\pi^{\left(n+\sum_{l=1}^{r}N_l\right)}}\frac{\partial^{N_1+N_2+\cdots+N_{r-1}+N_r-1}}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_{r-1}^{N_{r-1}}\partial x_r^{N_r-1}}$$
$$\cdot\sum_{v_1=0}^{q_1-1}\cdots\sum_{v_{r-1}=0}^{q_{r-1}-1}\sum_{v=0}^{n}\sum_{j=1}^{v+1}\frac{\Gamma(j+1)}{(1-t)^{j+1}}$$
$$\cdot\frac{b_v(k)c_{vj}[\beta(Z,Z)]^{-\left(n+\sum_{l=1}^{r}\frac{kN_l}{p_l}\right)}}{\left(1-\sum_{l=1}^{r-1}\omega_l^{v_l}x_l^{\frac{1}{q_l}}\right)^{v+1+\frac{1}{p_r}}},$$

where $|w_l|^2=|w_{l1}|^2+\cdots+|w_{lN_l}|^2$; $x_l=|w_l|^2\cdot[\beta(Z,Z)]^{-\frac{k}{p_l}}$, $1\leqslant l\leqslant r$; $q_l=\frac{1}{p_l}$, $\omega_l=e^{\frac{2\pi}{q_l}\sqrt{-1}}$, $1\leqslant l\leqslant r-1$; t is determined by (3). c_{vj}, $0\leqslant v\leqslant n$, $1\leqslant j\leqslant v+1$ are determined by (2). Let

$$f_{4k}(\lambda):=\frac{2^{n-1}(2k\lambda+n)\Gamma(k\lambda+n)}{\Gamma(k\lambda+1)},$$

then $b_0(k)=f_{4k}(-1)$,

$$b_j(k)=\frac{f_{4k}(-j-1)-\sum_{v=0}^{j-1}\frac{b_v(k)(-1)^v\Gamma(j+1)}{\Gamma(j-v+1)}}{(-1)^j\Gamma(j+1)},$$
$$j=1,2,\cdots,n.$$

Remark. When $k=1$, the results here are the same as the known results in Ref. [10].

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